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A Theorem on Weakly Regular Coequality Relation[1]

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Abstract

In preset paper we introduce and analyze a notion of weakly regular coequality relation on ordered set under an antiorder. For a coequality relation q on anti-ordered set $(X, =, \neq, \alpha)$ we say that it is a weakly regular coequality relation if $q^C \circ \alpha \subseteq \alpha \circ q^C$ holds. In that case, the factor-set $(X/q, =_1, \neq_1)$ is an ordered set under quasi-antiorder $\theta = \pi \circ \alpha \circ \pi^{-1}$. Besides, the natural mapping π is a strongly extensional isotone and reverse isotone embedding surjective function from $(X, =, \neq, \alpha)$ onto factor-set $(X/q, =_1, \neq_1, \theta)$.

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1 Introduction and preliminaries

This short investigation, in Bishop's constructive mathematics in sense of well-known books [1]-[3], [9] and Romano's papers [4]-[7], is continuation of forthcoming the second author's paper [8]. Since the Axioms System of the Constructive Logic is a part of the Axiom System of the Classical Logic, then mathematical development based on the Constructive Logic is acceptable in the Mathematics developed on the Classical Logic.

Let $(X, =, \neq)$ be a relational system, where the relation '$\neq$' is a binary relation on X, which satisfies the following properties:

$$\neg(x \neq x), \;\; x \neq y \Longrightarrow y \neq x, \;\; x \neq z \Longrightarrow x \neq y \vee y \neq z,$$
$$x \neq y \wedge y = z \Longrightarrow x \neq z$$

Follows Heyting, it is called *apartness.* A relation q on X is a *coequality relation* on X if and only if it is consistent, symmetric and cotransitive ([4]-[7]):

$$q \subseteq \neq, \;\; q^{-1} = q, \quad q \subseteq q * q,$$

where "$*$" is *filed product* between relations.

A relation α on X is *antiorder* ([4], [5]) on X if and only if

$$\alpha \subseteq \neq, \;\; \alpha \subseteq \alpha * \alpha, \neq \subseteq \alpha \cup \alpha \text{ (linearity)}$$

and a relation τ on X is a *quasi-antiorder* ([4]-[7]) on X if

$$\tau \subseteq \neq, \quad \tau \subseteq \tau * \tau.$$

Let x be an element of X and A a subset of X. We write $x \bowtie A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in X : x \bowtie A\}$.

If q is a coequality relation on a set X, then the relation $q^C = \{(x, y) \in X \times X : (x, y) \bowtie q\}$ is an equality on X compatible with q, in the following sense $q \circ q^C \subseteq q \wedge q^C \circ q \subseteq q$. We can construct factor-set $X/q = \{aq : a \in S\}$ with

$$aq =_1 bq \Longleftrightarrow (a, b) \bowtie q, \;\; aq \neq_1 bq \Longleftrightarrow (a, b) \in q.$$

The natural mapping $\pi : X \longrightarrow X/q$, defined by $\pi(a) =_1 aq$ for any $a \in X$, is a strongly extensional surjective function. It is easy to check that $q^C = \pi^{-1} \circ \pi$.

For a given anti-ordered set $(X, =, \neq, \alpha)$ is essential to know if there exists a coequality relation q on X such that X/q be an anti-ordered set. This plays an important role for studying the structure of anti-ordered sets. The following questions arise:
(1) Is there coequality relation q on X for which X/q is anti-ordered set?
(2) When the relation $\theta = \pi \circ \alpha \circ \pi^{-1}$ is an anti-order relation on X/q?
The concept of quasi-antiorder relation was introduced by the second author in his papers [4] and [5]. According to [5], if $(X, =, \neq, \alpha)$ is an anti-ordered set and σ a quasi-antiorder on X under α, then the relation q on X, defined by $q = \sigma \cup \sigma^{-1}$, is a coequality relation on X and the set X/q is an anti-ordered set under anti-order θ defined by $(xq, yq) \in \theta \Longleftrightarrow (x, y) \in \sigma$. So, according to results in [7], each quasi-antiorder σ on an ordered set X under anti-order α induces a coequality relation $q = \sigma \cup \sigma^{-1}$ on X such that X/q is an ordered set under antiorder θ. In paper [7] he proved that the converse of this statement also holds: If $(X, =, \neq, \alpha)$ is an anti-ordered set and q a coequality on X and if there exists an antiorder relation θ on X/q such that the $(X/q, =_1, \neq_1, \theta)$ is an ordered set under antiorder such that the mapping $\pi : X \longrightarrow X/q$ is a reverse isotone, then there exists a quasi-antiorder τ on X such that $q = \tau \cup \tau^{-1}$. (A function $f : (X, =, \neq, \alpha) \longrightarrow (Y, =, \neq, \beta)$ is an anti-order *reverse isotone* if $(f(a), f(b)) \in \beta \Longrightarrow (a, b) \in \alpha$ holds for any $a, b \in X$.) So, each coequality q on a set $(X, =, \neq, \alpha)$ such that X/q is an anti-ordered semigroup induces a quasi-antiorder on X. A coequality relation q on X is called *regular with respect to* α if there an antiorder "θ" on X/q satisfying the following conditions:
(i) $(X/q, =_1, \neq_1, \theta)$ is a anti-ordered set;
(ii) The mapping $\pi : X \ni a \longmapsto aq \in X/q$ is an anti-order reverse isotone function.
In article [6] he studied regular coequality relation on an anti-ordered set.

In the article [8] the second author studied *strongly regular coequality relation* q on $(X, =, \neq, \alpha)$, i.e. a regular coequality relation q with an additional condition: $q^C \circ \alpha \subseteq \alpha \circ q^C$.

In preset paper we introduce and analyze a new notion: For a coequality relation q on anti-ordered set $(X, =, \neq, \alpha)$ we say that it is a *weakly regular coequality relation* if $q^C \circ \alpha \subseteq \alpha \circ q^C$ holds. In that case, the factor-set $(X/q, =_1, \neq_1)$ is an ordered set under quasi-antiorder $\theta = \pi \circ \alpha \circ \pi^{-1}$. Besides, the natural mapping π is a strongly extensional isotone and reverse isotone embedding surjective function from $(X, =, \neq, \alpha)$ onto factor-set $(X/q, =_1, \neq_1, \theta)$.

2 The Results

In this paper we introduce and study a coequality relation q on an ordered set $(X, =, \neq)$ under an anti-order α when the following inclusion $q^C \circ \alpha \subseteq \alpha \circ q^C$ holds. As mention above, for such coequality we say that it is a *weakly regular coequality relation* on X. The following theorem is the main result of this paper.

Theorem 2.1 *: If the coequality relation q is a weakly regular, then the relation $\tau = \alpha \circ q^C$ is a quasi-antiorder relation on X and the factor-set $(X/q, =_1, \neq_1)$ is an ordered set under quasi-antiorder $\theta = \pi \circ \alpha \circ \pi^{-1}$ such that the mapping $\pi : X \longrightarrow X/q$ is a strongly extensional isotone and reverse isotone embedding and surjective function.*

Proof: (1) We have:
$\alpha \circ q^C \subseteq q^C \circ \alpha \circ q^C \subseteq q^C \circ (\alpha * \alpha) \circ q^C \subseteq (q^C \circ \alpha) * (\alpha \circ q^C)$
$\subseteq (\alpha \circ q^C) * (\alpha \circ q^C)$
because for any three relations $\alpha_1 \subseteq X_1 \times X_2$, $\alpha_2 \subseteq X_2 \times X_3$ and $\alpha_3 \subseteq X_3 \times X_4$

$$\alpha_3 * (\alpha_2 \circ \alpha_1) \supseteq (\alpha_3 * \alpha_2) \circ \alpha_1 \text{ and } (\alpha_3 \circ \alpha_2) * \alpha_1 \supseteq \alpha_3 \circ (\alpha_2 * \alpha_1)$$

are valid in the set $X_1 \times X_4$.
Indeed, let $a_1 \in X_1$ and $a_4 \in X_4$ such that $(a_1, a_4) \in (\alpha_3 * \alpha_2) \circ \alpha_1$. Then, there exists an element $a_2 \in X_2$ such that

$$(a_1, a_2) \in \alpha_1 \ \wedge \ (a_2, a_4) \in (\alpha_3 * \alpha_2)$$

and

$$(\exists a_2 \in X_2)((a_1, a_2) \in \alpha_1 \ \wedge \ (\forall z \in X_3)((a_2, z) \in \alpha_2 \ \vee \ (z, a_4) \in \alpha_3))).$$

Thus

$$(\exists a_2 \in X_2)(\forall z \in X_3)((((a_1, a_2) \in \alpha_1 \ \wedge \ (a_2, z) \in \alpha_2)) \vee ((a_1, a_2) \in \alpha_1 \ \wedge \ (z, a_4) \in \alpha_3))$$

and hence

$$(\forall z \in X_3)((((\exists a_2 \in X_2)((a_1, a_2) \in \alpha_1 \ \wedge \ (a_2, z) \in \alpha_2)) \vee ((a_1, a_2) \in \alpha_1 \ \wedge \ (z, a_4) \in \alpha_3)).$$

From above formula, we have

$$(\forall z \in X_3)((((\exists a_2 \in X_2)((a_1, a_2) \in \alpha_1 \ \wedge \ (a_2, z) \in \alpha_2)) \vee (z, a_4) \in \alpha_3).$$

The last is equivalent with the following

$$(\forall z \in X_3)((a_1, z) \in \alpha_2 \circ \alpha_1 \ \vee \ (z, a_4) \in \alpha_3)).$$

So, the last means

$$(a_1, a_4) \in \alpha_3 * (\alpha_2 \circ \alpha_1).$$

Analogously, we prove the following the inclusion

$$(\alpha_3 \circ \alpha_2) * \alpha_1 \supseteq \alpha_3 \circ (\alpha_2 * \alpha_1).$$

(2) Let us prove that the implication

$$q^C \circ \alpha \subseteq \alpha \circ q^C \Longrightarrow \alpha \circ q^C = q^C \circ \alpha \circ q^C$$

is valid. In fact:
(i) $\alpha \circ q^C = Id_X \circ \alpha \circ q^C \subseteq q^C \circ \alpha \circ q^C$;
(ii) $q^C \circ \alpha \circ q^C \subseteq \alpha \circ q^C \circ q^C \subseteq \alpha \circ q^C$.
Therefore, if the relation q is a weakly regular coequality relation on set $(X, =, \neq, \alpha)$, then holds $\alpha \circ q^C = q^C \circ \alpha \circ q^C$. Out of this, we conclude that the following relation

$$\pi \circ (\alpha \circ q^C) \circ \pi^{-1} = \pi \circ (\alpha \circ (\pi^{-1} \circ \pi)) \circ \pi^{-1} = (\pi \circ \alpha \circ \pi^{-1}) \circ (\pi \circ \pi^{-1}) = (\pi \circ \alpha \circ \pi^{-1}) = \theta$$

is a quasi-antiorder on $(X/q, =_1, \neq_1)$.

(3) Let a and b be arbitrary elements of X such that $(\pi(a), \pi(b)) \in \theta$. Thus, we have that
$$(a, b) \in \pi^{-1} \circ \theta \circ \pi = \pi^{-1} \circ (\pi \circ (\alpha \circ q^C) \circ \pi^{-1}) \circ \pi = (\pi^{-1} \circ \pi) \circ (\alpha \circ qC) \circ (\pi^{-1} \circ \pi)$$
$$= q^C \circ (\alpha \circ q^C) \circ q^C \subseteq \alpha \circ q^C \circ q^C \circ q^C \subseteq \alpha \circ q^C = \tau.$$
Opposite, if $(a, b) \in \alpha \subseteq \tau$, then
$$(\pi(a), \pi(b)) \in \pi \circ \tau \circ \pi^{-1} = \pi \circ \alpha \circ q^C \circ \pi^{-1} = \pi \circ \alpha \circ (\pi^{-1} \circ \pi) \circ \pi^{-1} = \pi \circ \alpha \circ \pi^{-1} = \theta.$$
So, the mapping π is a strongly extensional isotone and reverse isotone surjective function.

(4) Every isotone mapping $\pi : X \longrightarrow X/q$ satisfies the following condition: Let $x, y \in X$ and $x \neq y$. Then $(x, y) \in \alpha$ or $(y, x) \in \alpha$ by linearity of α and we have $(\pi(x), \pi(y)) \in \theta \subseteq \neq_1$ or $(\pi(y), \pi(x)) \in \theta \subseteq \neq_1$. So, the mapping is an embedding. Q.E.D.

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